

Chapter 2

Wave Equation

2.1 Scalar Wave Equation

In this lecture, we discuss principally on electromagnetic wave and elastic wave, because they are very important in many applications of measurement. These two waves are physically different, but mathematically they have a great similarity. It is the same for most of other wave phenomena, when it has linearity.

Mathematically, all the wave phenomena can be described by wave equation. For example, as the simplest example, scalar one-dimensional wave equation is represented as:

$$\nabla^2 u(x, t) = \frac{1}{v^2} \frac{\partial^2 u(x, t)}{\partial t^2} \quad (2.1.1)$$

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u(x, t)}{\partial t^2} \quad (2.1.2)$$

where $u(x, t)$ denotes a physical parameter, which is a function of time t and one-dimensional position x ,

∇^2 is the Laplacian operator to a scalar function and v is a constant value. The solution to (2.1.2) can be easily found and is given as:

$$u(x, t) = u^+(x - vt) + u^-(x + vt) \quad (2.1.3)$$

where $u^+(\cdot)$ and $u^-(\cdot)$ can be any scalar functions. The fact that the $u^+(\cdot)$ and $u^-(\cdot)$ are solution to (2.1.2) can be proven by putting (2.1.3) into (2.1.2). It should be noted that $u_1(x, t)$ and $u_2(x, t)$ denote physical quantities, which propagate to the positive and negative x -directions at the constant velocity of v .

When some boundary conditions are implied to the wave equation, $u^+(\cdot)$ and $u^-(\cdot)$ cannot be arbitrary functions any more, and have to be determined by the boundary conditions. This is the case for the most physical wave phenomena. The boundary conditions include the condition of the medium, in which the wave propagate and the method of excitation of the wave motion.

If the $u_1(x, t)$ and $u_2(x, t)$ are the solution to the wave equation, which satisfies the boundary conditions, then their sum:

$$u(x, t) = u_1(x, t) + u_2(x, t) \quad (2.1.4)$$

is also the solution to the wave equation. Using the linear properties of the quantities can prove this fact.

2.2 Time harmonic solution to the wave equation

Now we consider solving the wave equation for time-harmonic quantities. Now we assume a phaser notation of

$u(x, t)$ as $U(x)$ and simply write as U . Substituting U and using the conditions defined by (1.2.3) and (1.2.4), we have

$$\nabla^2 U(x) = -\frac{\omega^2}{v^2} U(x) \quad (2.2.1)$$

Assuming $U(x) = e^{\lambda x}$ as a solution, where λ is an unknown constant value, and substituting $U(x) = e^{\lambda x}$ into (2.2.1) we have:

$$\lambda^2 = -\frac{\omega^2}{v^2} \quad (2.2.2)$$

This equation determines the unknown value λ . (2.2.2) is referred to as a dispersion equation. And

$$k = \pm j\lambda = \pm \frac{\omega}{v} \quad (2.2.3)$$

is a wave number. By using the wave number, the time-harmonic solution to the wave equation (2.2.1) is given by:

$$U(x) = U_1 e^{-jkx} + U_2 e^{+jkx} \quad (2.2.4)$$

where U_1 and U_2 are constant values, which will be determined by the boundary conditions.

2.3 The diffusion equation

In order to compare the wave equation to another differential equation, which is governing the transfer phenomena in physics, now we consider a heat flow along an insulated bar. Temperature at each position x and time t is $u(x, t)$. Define the linear density of the material be ρ and the specific heat of the material: be c then the amount of heat energy in the portion of the bar in $a \leq x \leq b$ is given as:

$$\int_a^b u(x, t) c(x) \rho(x) dx \quad (2.3.1)$$

The rate of change of the heat energy is given by:

$$\frac{d}{dt} \int_a^b u(x, t) c(x) \rho(x) dx \quad (2.3.2)$$

and we have:

$$\frac{d}{dt} \int_a^b u(x, t) c(x) \rho(x) dx = F(a, t) - F(b, t) \quad (2.3.3)$$

where $F(x, t)$ is the rate at which heat energy passes the point. The above equation can be rewritten as:

$$\int_a^b \frac{\partial u}{\partial t}(x,t)c(x)\rho(x) + \frac{\partial F(x,t)}{\partial x} dx = 0 \quad (2.3.4)$$

and

$$\frac{\partial u}{\partial t}(x,t)c(x)\rho(x) + \frac{\partial F(x,t)}{\partial x} = 0 \quad (2.3.5)$$

have to be satisfied. The heat flux should depend on the spatial rate of change of the temperature. This is expressed in Fourier's law of cooling:

$$F(x,t) = -\kappa \frac{\partial u(x,t)}{\partial x} \quad (2.3.6)$$

where κ is the heat conductivity. Substituting (6) into (5) we have

$$\frac{\partial u(x,t)}{\partial t}c(x)\rho(x) = \frac{\partial^2 \kappa u(x,t)}{\partial x^2} \quad (2.3.7)$$

when the material constants are independent on the position x , (7) yields:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = k \frac{\partial u(x,t)}{\partial t} \quad (2.3.8)$$

where $k = \frac{c\rho}{\kappa}$ This is a one-dimensional heat equation.

The analogue of Fourier's law of cooling is known as Fick's law of diffusion which states that the flux is proportional to the spatial ratio of change of the concentration, and that dye moves from region of higher concentration to region of lower concentration This leads to diffusion equation. Compare (2.3.8) with (2.1.2), i.e., wave equation.